A SEQUENCE OF PROBLEMS FOR CALCULUS

J. W. NEUBERGER

Problem 1. *Suppose that* $f(x) = x^2$ *for all numbers* x. *If* a *is a number, determine what*

$$\frac{f(p) - f(a)}{p - a}$$

approaches as p *approaches* a.

CLASS

Problem 2. *Suppose that* $f(x) = x^3$ *for all numbers* x. *If* a *is a number, determine what*

$$\frac{f(p) - f(a)}{p - a}$$

approaches as p *approaches* a.

CLASS

Problem 3. *Suppose that* $f(x) = x^4$ *for all numbers* x. *If* a *is a number, determine what*

$$\frac{f(p) - f(a)}{p - a}$$

approaches as p *approaches* a.

CLASS

Problem 4. *Suppose that* $f(x) = x^5$ *for all numbers* x. *If* a *is a number, determine what*

$$\frac{f(p) - f(a)}{p - a}$$

approaches as p *approaches* a.

Garcia

Problem 5. *Suppose that* $f(x) = \frac{1}{x}$ *for all* $x \neq 0$. *If* $a \neq 0$ *determine what*

$$\frac{f(p) - f(a)}{p - a}$$

approaches as p *approaches* a.

T. Green, A. Sivakumaran

Date: 24 November 2003.

Problem 6. *Suppose that* $f(x) = \frac{1}{x^2}$ *for all* $x \neq 0$*. If* $a \neq 0$ *determine what*

$$\frac{f(p) - f(a)}{p - a}$$

approaches as p *approaches* a*.*

S. Carpenter

Problem 7. *Suppose that* $f(x) = x^{\frac{1}{2}}$ *for all* $x > 0$*. If* $a > 0$*, determine what*

$$\frac{f(p) - f(a)}{p - a}$$

approaches as p *approaches* a*.*

Kim

Problem 8. *Suppose that* $f(x) = x^{\frac{1}{3}}$ *for all* $x > 0$*. If* $a > 0$*, determine what*

$$\frac{f(p) - f(a)}{p - a}$$

approaches as p *approaches* a*.*

Morris, Reardon, Hermanson

Problem 9. *Suppose that* $f(x) = x^{\frac{2}{3}}$ *for all* $x > 0$*. If* $a > 0$*, determine what*

$$\frac{f(p) - f(a)}{p - a}$$

approaches as p *approaches* a*.*

Sivakumaran

Problem 10. *Suppose that* $f(x) = \frac{1}{x^{\frac{1}{2}}}$ *for all* $x > 0$*. If* $a > 0$*, determine what*

$$\frac{f(p) - f(a)}{p - a}$$

approaches as p *approaches* a*.*

Crawford

Problem 11. *Suppose that* $f(x) = x^4 + \frac{1}{x}$ *for all* $x \neq 0$*. If* $a > 0$*, determine what*

$$\frac{f(p) - f(a)}{p - a}$$

approaches as p *approaches* a*.*

Casey, Weeks

Problem 12. *Given numbers* p, q *and a positive integer* n*, show how to factor*

$$p^n - a^n \;=\; (p - a) \times (?).$$

Suppose that n is a positive integer and $f(x) = x^n$ for all numbers x. If a is a number, determine what

$$\frac{f(p) - f(a)}{p - a}$$

approaches as p approaches a.

Wong

Problem 13. *Suppose that n is a positive integer and $f(x) = x^{\frac{1}{n}}$ for all numbers x. If $a > 0$, determine what*

$$\frac{f(p) - f(a)}{p - a}$$

approaches as p approaches a. What happens if $a = 0$.

Pickens

Problem 14. *Suppose that n is a positive integer and $f(x) = \frac{1}{x^n}$ for all $x \neq 0$. If $a \neq 0$ determine what*

$$\frac{f(p) - f(a)}{p - a}$$

approaches as p approaches a.

Kim

Notation If f is a function and there is at least one number a in the domain of f so that that

$$\frac{f(p) - f(a)}{p - a}$$

approaches some number as p approaches a, then f' denotes the function (whose domain is all such numbers a) such that

$$\frac{f(p) - f(a)}{p - a} \text{ approaches } f'(a) \text{ as } p \text{ approaches } a.$$

From here on we write the above line as

$$\frac{f(p) - f(a)}{p - a} \to f'(a) \text{ as } p \to a.$$

Problem 15. *Suppose that n is a positive integer and f is the function so that*

$$f(x) = \frac{1}{x^{1/n}}$$

for all numbers $x > 0$. Find f'.

Pickens

Problem 16. *Suppose that each of g and h is a function with domain all numbers and f is the function so that*

$$f(x) \ = \ g(x) + h(x)$$

Suppose also that each of g' and h' has domain all numbers. Show that

$$f'(x) \ = \ g'(x) + h'(x)$$

for all numbers x.

Hoskinson

Problem 17. *Suppose that $f(x) \ = \ (x^2+x+1)(x^3+x^2+x+1)$ for all numbers x. Find f'.*

Baldwin

Problem 18. *Suppose that each of g and h is a function with domain all numbers so that each of g' and h' have domain all numbers. Suppose also that f is the function so that $f(x) = g(x)h(x)$ for all numbers x (that is, $f = gh$). Show first that if p and a are two numbers then*

$$\frac{f(p)-f(a)}{p-a} \ = \ g(a)\frac{h(p)-h(a)}{p-a} + \frac{g(p)-g(a)}{p-a}h(a)$$

and secondly show that if x is any number then

$$f'(x) = g(x)h'(x) \ + g'(x)h(x).$$

Hoskinson

Problem 19. *Suppose that g is a function with domain all numbers so that g' also has domain all numbers. Suppose in addition that c is a number and f is the function so that*

$$f(x) \ = \ cg(x)$$

for all numbers x. Find f'.

Problem 20. *Suppose that each of f, g and h is a function with domain all numbers and f is the function so that*

$$f(x) = (6x^7 + 4x)^{\frac{1}{3}}$$

for all numbers x. Find functions g and h so that

$$f(x) \ = \ h(g(x))$$

for all numbers x. Find f' in terms of g and h.

Problem 21. *Suppose that each of g and h is a function with domain all numbers so that g' and h' also have domain all numbers. Suppose in addition that f is the function so that*

$$f(x) \ = \ g(h(x))$$

for all numbers x. Show that if a and p are numbers, and $h(p) \neq h(a)$, then

$$\frac{f(p)-f(a)}{p-a} = \frac{g(h(p))-g(h(a))}{h(p)-h(a)}\frac{h(p)-h(a)}{p-a}.$$

and that for each number x,

$$f'(x) = g(x)h'(x) + g'(x)h(x).$$

Obioha

Problem 22. *Suppose*

$$f(x) = x^{\frac{1}{3}}(14x^3 + 7x^2 - 4)$$

for all numbers x for which this is defined. Find f'.

Mrnustik

Problem 23. *Suppose*

$$f(x) = (4x^3 + 3x^2 - 5x + 1)^{\frac{1}{5}}(6x^2 - 3x + 7)^{-\frac{1}{3}}$$

for all numbers x for which this is defined. Find f'.

Sorrells, Carpenter

Problem 24. *Suppose h is a function which has a derivative and f is the function so that*

$$f(x) = \frac{1}{h(x)}$$

for all numbers x is the domain of h so that $h(x) \neq 0$.

(a) find a function g so that

$$f(x) = g(h(x))$$

for all numbers x in the domain of f and

(b) find f'.

Tallman

Problem 25. *Suppose that each of g and h is a function with domain all numbers so that g' and h' each have domain all numbers also. Suppose in addition that $h(x) \neq 0$ for all numbers x. If*

$$f(x) = \frac{g(x)}{h(x)}$$

for all numbers x, show that

$$f'(x) = \frac{g'(x)h(x) - g(x)h'(x)}{h(x)^2}$$

for all numbers x.

Reardon,Black

Problem 26. *Suppose that*

$$f(x) = \frac{x^2 - 2}{x^2 + 1}$$

for all numbers x. Find f'.

Hermanson

Problem 27. *Suppose that*

$$f(x) = \frac{x^3 + 3x^2 - 7x + 4}{x^2 + x + 1}$$

for all numbers x. Find f'.

Weeks

Problem 28. *Find a function f so that*

$$f'(x) = x^2$$

for all numbers x

Pickens

Problem 29. *Find a function f so that*

$$f'(x) = x^3$$

for all numbers x

Snow

Problem 30. *Find a function f so that*

$$f'(x) = \frac{1}{x^2}$$

for all numbers $x \neq 0$.

Russell

Problem 31. *Find a function f so that*

$$f'(x) = (x^2 + x + 1)^3(2x + 1)$$

for all numbers x

Hoskinson

Problem 32. *Suppose n is a positive integer no smaller than 2. Find a function f so that*

$$f'(x) = \frac{1}{x^n}$$

for all numbers $x \neq 0$.

Crawford

Problem 33. *Can you think of a reason why, in Problem 32, it was asked that the positive integer n be no smaller than 2?*

class

Problem 34. *Suppose that $f(x) = \sin(x)$ for all numbers x, the sine function taken with respect to radian measure. Find $f'(0)$.*

Hint: Take a number x between 0 and $\pi/2$. Draw two triangles, one with verticies $(0,0), (\cos(x), 0), (\cos(x), \sin(x))$ and the other with verticies $(0,0), (1,0), (1, \tan(x))$. Find the area of these triangles and also the area of the sector defined by (a) the interval from $(0,0)$ to $(1,0)$, (b) the arc of the unit circle from $(1,0)$ to $(\cos(x), \sin(x))$ and (c) the interval from $(0,0)$ to $(\cos(x), \sin(x))$. Make an inequality from these three numbers. See what information this gives concerning

$$\frac{\sin(x) - sin(0)}{x - 0}.$$

Wong

Problem 35. *Find a function f so that*

$$f'(x) = x^{\frac{1}{2}}$$

for all $x > 0$ and find a function f so that

$$f'(x) = x^{-\frac{1}{2}}$$

for all $x > 0$.

Short

Problem 36. *Suppose*

$$f(x) = (x^2 + (4x^2 + 2x^2 - x + 1)^{\frac{1}{3}})^{-\frac{1}{3}}$$

for all x for which this is defined. Find f'.

Baldwin

Problem 37. *Suppose*

$$f(x) = \frac{(7x^4 - 6x^2 + 1)^{\frac{1}{3}}}{(5x^5 + x^2 + 7)^{\frac{1}{4}}}$$

for a x for which this is defined. Find f'.

Christian

Problem 38. *Find a function f so that*

$$f'(x) = (x^3 + 3x^2 - 4x + 1)^{\frac{1}{3}}(x^2 + 2x - \frac{4}{3})$$

for a x for which this is defined.

Hoskinson

Problem 39. *Suppose $f(x) = sin(x)$ for all numbers x. Find f'.*

Recall that for each of x and y a number,

$$\sin(x + y) = \sin(x)\cos(x) + \cos(x)\sin(x).$$

Goldenburg, Black

Problem 40. *Suppose $f(x) = \cos(x)$ for all numbers x. Find f'.*

Recall that

$$\cos(x) = \sin(x + \frac{\pi}{2})$$

for all numbers x.

Baldwin

Problem 41. *Find derivatives for all six of the trigonometric functions.*

Black, Green, Muno, Sivakumaran

Problem 42. *Suppose that*

$$f(x) = Sin^{-1}(x)$$

for all numbers x in $[-1, 1]$. Find f'.

Wang, Goldenburg

Problem 43. *Suppose that $f(x) = 1/x$ for all $x > 0$. Find a number which is within .01 of the area of the region bounded by the X-axis, the graph of f and the vertical line through the point $(1, 0)$ and the vertical line through the point $(2, 0)$. (Recall the definition of upper sum and lower sum).*

Carpenter

Problem 44. *Find a function f so that $f'(x) = x(3x^2 + 1)^{\frac{1}{2}}$ for all numbers x.*

Wohlmuth

Problem 45. *Find a function f so that $f'(x) = \frac{x}{3x^2+1)^2}$ for all numbers x.*

Grant

Problem 46. *Find a function f so that $f'(x) = x(sin(x^2))$ for all numbers x.*

Rupley

Problem 47. *Suppose that $f(x) = Tan^{-1}x$ for all numbers x. Find f'.*

Hoskinson

Problem 48. *Suppose that $f(x) = Sec^{-1}x$ for all numbers x for which this is defined. Find f'.*

Crawford

Problem 49. *For L as in Problem 43, show that $L'(x) = f(x)$ for all $x > 0$.*

Worcester

Problem 50. *Find good approximations to the number e so that $L(e) = 1$.*

Problem 51. *Find a function f so that*

$$f'(x) = \frac{1}{(1 - 4x^2)^{\frac{1}{2}}}$$

for all numbers x for which this is defined.

class

Problem 52. *Find a function f so that*

$$f'(x) = \frac{1}{x^2 + x + 1}$$

for all numbers x.

Hoskinson

Problem 53. *Find a function f so that*

$$f'(x) = \frac{1}{1 - x^2}$$

for all numbers x for which this is defined.

Green

Problem 54. *Find a function f so that*

$$f'(x) = \frac{1}{x^2 - 5x + 6}$$

for all numbers x for which this is defined.
Baldwin

Problem 55. *Find a function f so that*

$$f'(x) = \frac{2x - 5}{x^2 - 5x + 6}$$

for all numbers x for which this is defined.

Green

Problem 56. *Find a function f so that*

$$f'(x) = \frac{x}{x^2 - 5x + 6}$$

for all numbers x for which this is defined.

Mrnustik

Problem 57. *A rectangular pasture is to have area 2 with one side on a river (which does not require a fence). Find the dimensions of such a pasture which uses a minimum amount of fence.*

Worcester

Problem 58. *Find a function f so that*

$$f'(x) = \frac{x^5}{x^2 + 1}$$

for all numbers x.

Baldwin

Find a function f so that

$$f'(x) = \frac{x^5 + sx^2 + 1}{x^2 + 1}$$

Baldwin

Problem 59. *Find a function f so that*

$$f'(x) = \frac{x^2}{x^2 + x + 1}$$

for all numbers x.

Carpenter

Problem 60. *Find a function f so that*

$$f'(x) = \frac{1}{(1 + x - x^2)^{\frac{1}{2}}}$$

for all numbers x for which this is defined.

Hoskinson, Carpenter

Problem 61. *Find a function f so that*

$$f'(x) = \frac{1}{x^2 + x + \frac{1}{10}}$$

for all numbers x for which this is defined.

Worcester

Problem 62. *Suppose that $E = L^{-1}$. Find E'.*

class

Problem 63. *Suppose that*

$$f(x) = \frac{1}{x^2}$$

for all x so that $1 \leq x \leq 2$. Find the volume of the figure obtained by rotating the graph of f about the X axis.

Green

Problem 64. *Suppose that*

$$f(x) = \sin(x)$$

for all x so that $0 \leq x \leq \pi$. Find the volume of the figure obtained by rotating the graph of f about the X axis.

Green

Problem 65. *Find a function f so that*

$$f'(x) = (\cos(x))^2$$

for all numbers x.

Problem 66. *Find a function f so that*

$$f'(x) = x \ln(x)$$

for all numbers $x > 0$.

Problem 67. *Find a function f so that*

$$f'(x) = x\sin(x)$$

for all numbers x.

Problem 68. *Suppose f*

$$f(x) = |x|^{\frac{1}{2}}$$

for all numbers x. Find the minimum of f.

Problem 69. *Suppose f is the function so that*

$$f(x) = \sin(\frac{1}{x})$$

for all $x > 0$. Plot the graph of f.

Theorem. Suppose $[a, b]$ is an interval and f is a function so that f has a derivative at each point of $[a, b]$. Then there is a number c in $[a, b]$ so that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Problem 70. *Find a function f so that*

$$f'(x) = \frac{1}{x(1-x)^2}$$

for all numbers x for which this is defined.

Problem 71. *Find a function f so that*

$$f'(x) = \frac{1}{x(x^2+1}$$

for all numbers x..

Problem 72. *Show that*

$$L(4) - L(2) = L(2)$$

and

$$L(8) - L(4) = L(2).$$

(Use upper and lower sums).

Problem 73. *Suppose that*

$$f(x) = \frac{x}{1+x^2}$$

for all numbers x. Find the area beneath the graph of f from 2 to 4.

Problem 74. *Suppose that*

$$f(x) = \frac{1}{1+x+x^2}$$

for all numbers x. Find the area beneath the graph of f from -1 to 1.

Problem 75. *Suppose that*

$$f(x) = x + 1$$

for all numbers x*. Find the area beneath the graph of* f *from* 0 *to* 3*.*

Problem 76. *Find a function* f *so that*

$$f'(x) = f(x)$$

and $f(x) \neq x$ *for all numbers* x*.*

Department of Mathematics
University of North Texas
Denton, TX 76203
E-mail address: jwn@unt.edu